2023-24 MATH2048: Honours Linear Algebra II

Homework 2 Solution

Due: 2023-09-22 (Friday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let V and W be vector spaces over the field F, and let V_1 and W_1 be subsets of V and W respectively. Consider the direct product $V \times W$.

   Prove or disprove: If V_1 is a subspace of V and W_1 is a subspace of W, then $V_1 \times W_1$ is a subspace of $V \times W$.

   Prove or disprove: If the product set $V_1 \times W_1$ is a subspace of $V \times W$, then V_1 is a subspace of V and W_1 is a subspace of W.

Proof. (a) Assume V_1 is a subspace of V and W_1 is a subspace of W.

Because both V_1 and W_1 contain the zero vector, we have $(0,0) \in V_1 \times W_1$.

For any $a, b \in V_1 \times W_1$ and scalar $k \in F$, we have $a = (v_1, w_1), b = (v_2, w_2)$ for some $v_1, v_2 \in V_1$ and $w_1, w_2 \in W_1$. Then $v_1 + kv_2 \in V_1$ and $w_1 + kw_2 \in W_1$. Then $a + kb = (v_1 + kv_2, w_1 + kw_2) \in V_1 \times W_1$. Hence, $V_1 \times W_1$ is a subspace of $V \times W$.

(b) Assume $V_1 \times W_1$ is a subspace of $V \times W$. Since $(0,0) \in V_1 \times W_1$, it follows that $0 \in V_1$ and $0 \in W_1$, so both V_1 and W_1 are nonempty.

Now, let's take any $a, b \in V_1$ and $c, d \in W_1$, and a scalar $k \in F$. Because $(a,c), (b,d) \in V_1 \times W_1$ and $V_1 \times W_1$ is a subspace, we know that $(a+kb, c+kd) \in V_1 \times W_1$.

By the definition of the Cartesian product, this implies that $a + kb \in V_1$ and $c + kd \in W_1$. This shows that V_1 and W_1 are closed under addition and scalar multiplication, and hence, they are subspaces of V and W, respectively.

□

2. Let V be a finite dimensional vector space and W be a subspace of V. Define a map $\pi : V \to V/W$ by $\pi(v) = v + W$ for all $v \in V$. Show that π is a surjective linear transformation and its kernel is W.

Proof. Let's prove that π is a surjective linear transformation and its kernel is W.

Firstly, let's prove that π is a linear transformation. Let $v_1, v_2 \in V$ and $k \in F$.

We have $\pi(v_1 + v_2) = v_1 + v_2 + W = (v_1 + W) + (v_2 + W) = \pi(v_1) + \pi(v_2)$ and $\pi(kv_1) = kv_1 + W = k(v_1 + W) = k\pi(v_1)$. Thus, π is a linear transformation.

Secondly, for all $w \in V/W$, $w = v + W$ for some $v \in V$. Then $\pi(v) = w$, which shows surjectivity.

Finally, $\ker(\pi) = \{v \in V : \pi(v) = 0 + W\} = \{v \in V : v + W = 0 + W\} = W$. □

3. Let $\{v_i\}_{i\in I}$ be a spanning set of a (maybe infinite-dimensional) vector space V. Prove that there exists a subset $S \subseteq I$ such that $\{v_i\}_{i\in S}$ is a basis of V. (Hint: Use Zorn's lemma to prove a maximal S exists.)

Proof. To prove that every vector space has a basis, we will use Zorn's Lemma. Zorn's Lemma states that if every chain (a totally ordered subset) in a partially ordered set has an upper bound, then the set contains at least one maximal element.

First, consider the set S of all linearly independent subsets of a given vector space V over a field F. We order this set by inclusion.

Now we need to show that every chain in S has an upper bound in S. Let C be a chain in S, which is a collection of linearly independent subsets of V ordered by inclusion. Let U be the union of all the sets in C. We claim that U is a linearly independent set. Suppose there exist vectors $v_1, v_2, ..., v_n \in U$ and scalars $a_1, a_2, ..., a_n \in F$, with $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$.

However, each v_i is in some set in the chain C, and since C is totally ordered by inclusion, there is one set in C that contains all of the vectors $v_1, v_2, ..., v_n$. But this set is in S and is therefore linearly independent. Then each a_i is zero.

Therefore, U is a linearly independent set and is in S. So, every chain in S has an upper bound in S.

By Zorn's Lemma, S contains at least one maximal element. Call it B. This is a linearly independent set that is not properly contained within any other linearly independent set in V.

Now we need to show that B spans V. Assume for contradiction that it does not. Then there exists a vector $v \in V$ that is not in the span of B. We can add v to B to form a larger linearly independent set, contradicting the maximality of B. Therefore, the assumption that B does not span V must be false.

Hence, B is a basis for V. □

4. (2.1 Q20) Let V and W be vector spaces with subspaces V_1 and W_1, respectively. If $T : V \to W$ is linear, prove that $T(V_1)$ is a subspace of W and that $\{x \in V : T(x) \in W_1\}$ is a subspace of V.

Proof. First, let's prove that $T(V_1)$ is a subspace of W.

(a) $T(0) = 0 \in T(V_1)$.

(b) Let $w_1 = T(v_1)$ and $w_2 = T(v_2)$ be any two vectors in $T(V_1)$, for some $v_1, v_2 \in V_1$. Then, $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$, which is in $T(V_1)$ since $v_1 + v_2 \in V_1$.

(c) Let $w = T(v)$ be any vector in $T(V_1)$, for some $v \in V_1$, and let k be any scalar. Then, $kw = kT(v) = T(kv)$, which is in $T(V_1)$ since $kv \in V_1$.

Thus, $T(V_1)$ is a subspace of W.

Second, let's prove that $S = \{x \in V : T(x) \in W_1\}$ is a subspace of V.

(a) Because $T(0) = 0 \in W_1$, $0 \in S$.

(b) Let $x_1, x_2 \in S$. Then, $T(x_1 + x_2) = T(x_1) + T(x_2) \in W_1$ because W_1 is a subspace of W. Therefore, $x_1 + x_2 \in S$.

(c) Let $x \in S$ and k be any scalar. Then, $T(kx) = kT(x) \in W_1$ because W_1 is a subspace of W. Therefore, $kx \in S$.

Thus, S is a subspace of V.

□

5. (2.1 Q13) Let V and W be vector spaces, let $T : V \to W$ be linear, and let $\{w_1, w_2, ..., w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, ..., v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, ..., k$, then S is linearly independent.

Proof. Suppose there exist scalars $a_1, a_2, ..., a_k$, such that $a_1v_1 + a_2v_2 + ... + a_kv_k = 0$. Then $T(a_1v_1 + a_2v_2 + ... + a_kv_k) = a_1w_1 + a_2w_2 + ... + a_kw_k = 0$. By the linear independence of $w_1, w_2, ..., w_k$, each $a_i = 0$. Hence, S must be linearly independent.

$\square$